

Constructing multi-player quantum games from non-factorizable joint probabilities

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Abstract

We use the standard three-party Einstein-Podolsky-Rosen (EPR) setting in order to play general three-player non-cooperative symmetric games. We analyze how the peculiar non-factorizable joint probabilities that may emerge in the EPR setting can change outcome of the game. Our setup requires that the quantum game attains classical interpretation for factorizable joint probabilities. We analyze the generalized three-player game of Prisoner's Dilemma (PD) and show that the players can indeed escape from the classical outcome of the game because of non-factorizable joint probabilities. This result for three-player PD contrasts strikingly with our earlier result for two-player PD for which even non-factorizable joint probabilities are not found to be helpful to escape from the classical outcome of the game.

1 Introduction

The usual approach to quantum games [1] considers an initial (entangled!) quantum state on which players perform local actions (strategies) and the state evolves to the final state. Payoffs are generated in the last step involving quantum measurement on the final state. This approach assumes familiarity with the concepts of product or entangled quantum state(s), expectation values, trace operation, density operators, and the theory of quantum measurement.

This paper presents a probabilistic approach to quantum games that constructs true quantum games from probabilities only. Our motivation has been to present quantum game to the wider audience, especially to those readers who use elements of game theory but find the concepts of quantum mechanics rather alien. We use probabilities to construct quantum games because, after all, Bell

inequalities [2] can also be understood in this way, widely believed to express the true quantum behavior.

We extend our probabilistic framework [3] for two-player quantum games to mutliplayer case, while using non-factorizable joint probabilities to construct quantum games. Apart from opening quantum games to the readers outside of the quantum physics, this framework provides a unifying perspective on both the classical and the quantum games.

2 Three-player, two-strategy, non-cooperative, symmetric games

We consider three-player symmetric games for which players' pure strategies are given as Alice: S_1, S_2 ; Bob: S'_1, S'_2 ; Chris: S''_1, S''_2 , and players' payoff relations are

$$\begin{aligned} \Pi_{A,B,C}(S_1, S'_1, S''_1) &= \alpha, \alpha, \alpha; & \Pi_{A,B,C}(S_1, S'_2, S''_2) &= \epsilon, \theta, \theta; \\ \Pi_{A,B,C}(S_2, S'_1, S''_1) &= \beta, \delta, \delta; & \Pi_{A,B,C}(S_2, S'_1, S''_2) &= \theta, \epsilon, \theta; \\ \Pi_{A,B,C}(S_1, S'_2, S''_1) &= \delta, \beta, \delta; & \Pi_{A,B,C}(S_2, S'_2, S''_1) &= \theta, \theta, \epsilon; \\ \Pi_{A,B,C}(S_1, S'_1, S''_2) &= \delta, \delta, \beta; & \Pi_{A,B,C}(S_2, S'_2, S''_2) &= \omega, \omega, \omega, \end{aligned} \quad (1)$$

where the subscripts refer to the players, the three entries in braces on left side are pure strategies of Alice, Bob, and Chris, respectively, and the three entries on right are their payoffs. The three-player Prisoners' Dilemma offers an example of such a game.

2.1 Three-player Prisoners' Dilemma

In this game each of the three players Alice, Bob, and Chris has two pure strategies: C (Cooperation) and D (Defection). Using the notation introduced in (1) we associate

$$\text{Alice: } S_1 \sim C, S_2 \sim D; \text{ Bob: } S'_1 \sim C, S'_2 \sim D; \text{ Chris: } S''_1 \sim C, S''_2 \sim D. \quad (2)$$

The three-player Prisoners' Dilemma [4] is defined by requiring that S_2 is a dominant choice for each player:

$$\begin{aligned} \Pi_A(S_2, S'_1, S''_1) &> \Pi_A(S_1, S'_1, S''_1), \\ \Pi_A(S_2, S'_2, S''_2) &> \Pi_A(S_1, S'_2, S''_2), \\ \Pi_A(S_2, S'_1, S''_2) &> \Pi_A(S_1, S'_1, S''_2), \end{aligned} \quad (3)$$

and similar inequalities hold for players Bob and Chris. Secondly, a player is better off if more of his opponents choose to cooperate:

$$\begin{aligned} \Pi_A(S_2, S'_1, S''_1) &> \Pi_A(S_2, S'_1, S''_2) > \Pi_A(S_2, S'_2, S''_2), \\ \Pi_A(S_1, S'_1, S''_1) &> \Pi_A(S_1, S'_1, S''_2) > \Pi_A(S_1, S'_2, S''_2). \end{aligned} \quad (4)$$

Thirdly, if one player's choice is fixed, the other two players are left in the situation of a two-player PD:

$$\begin{aligned}
\Pi_A(S_1, S'_1, S''_2) &> \Pi_A(S_2, S'_2, S''_2), \\
\Pi_A(S_1, S'_1, S''_1) &> \Pi_A(S_2, S'_1, S''_2), \\
\Pi_A(S_1, S'_1, S''_2) &> (1/2) \{ \Pi_A(S_1, S'_2, S''_2) + \Pi_A(S_2, S'_1, S''_2) \}, \\
\Pi_A(S_1, S'_1, S''_1) &> (1/2) \{ \Pi_A(S_1, S'_1, S''_2) + \Pi_A(S_2, S'_1, S''_1) \}.
\end{aligned} \tag{5}$$

Using the notation (1) these conditions require

$$\begin{aligned}
\text{a) } &\beta > \alpha, \quad \omega > \epsilon, \quad \theta > \delta \\
\text{b) } &\beta > \theta > \omega, \quad \alpha > \delta > \epsilon \\
\text{c) } &\delta > \omega, \quad \alpha > \theta, \quad \delta > (1/2)(\epsilon + \theta), \quad \alpha > (1/2)(\delta + \beta)
\end{aligned} \tag{6}$$

3 Playing three-player games using coins

The above game can be played using coins and in the following we consider two setups to achieve this.

3.1 Three-coin setup

This setup involves sharing three coins among Alice, Bob, and Chris. We define pure strategies by the association:

$$S_1, S'_1, S''_1 \sim \text{flip}, \quad S_2, S'_2, S''_2 \sim \text{do not flip}, \quad \text{Head} \sim +1, \quad \text{Tail} \sim -1, \tag{7}$$

and play the game as follows. In a run each player receives a coin in 'head up' state which s/he can 'flip' or 'does not flip'. After players' actions the coins are passed to a referee. The referee observes the coins and rewards the players. A player can play a mixed strategy (definable for many runs) by flipping his/her coin with some probability. This allows us to write mixed strategies as (x, y, z) where x, y , and z are the probabilities with which Alice, Bob, and Chris flip their coins, respectively. The mixed-strategy payoff relations read

$$\begin{aligned}
\Pi_{A,B,C}(x, y, z) = &xyz(\alpha, \alpha, \alpha) + x(1-y)z(\delta, \beta, \delta) + xy(1-z)(\delta, \delta, \beta) + \\
&x(1-y)(1-z)(\epsilon, \theta, \theta) + (1-x)yz(\beta, \delta, \delta) + (1-x)(1-y)z(\theta, \theta, \epsilon) + \\
&(1-x)y(1-z)(\theta, \epsilon, \theta) + (1-x)(1-y)(1-z)(\omega, \omega, \omega)
\end{aligned} \tag{8}$$

Assuming (x^*, y^*, z^*) to be a Nash equilibrium (NE) requires:

$$\begin{aligned}
\Pi_A(x^*, y^*, z^*) - \Pi_A(x, y^*, z^*) &\geq 0, \\
\Pi_B(x^*, y^*, z^*) - \Pi_B(x^*, y, z^*) &\geq 0, \\
\Pi_C(x^*, y^*, z^*) - \Pi_B(x^*, y^*, z) &\geq 0.
\end{aligned} \tag{9}$$

3.2 Six-coin setup

This setup translates playing of a three-player game in terms of joint probabilities which may attain the unusual character of being non-factorizable for certain quantum systems. The setup thus allows how non-factorizable (quantum) probabilities lead to game-theoretic consequences.

In this setup, each player receives two coins (either one can be in head or tail state) and each player chooses one out of the two coins given to him/her. Players pass three chosen coins to the referee who tosses the three chosen coins and observes the outcome. After many runs the referee rewards the players.

Players' strategies are defined by establishing the association:

$$S_1, S'_1, S''_1 \sim \text{choose the first coin}, \quad S_2, S'_2, S''_2 \sim \text{choose the second coin} \quad (10)$$

A player plays a pure strategy when h/she chooses the same coin for all runs. He/she plays a mixed-strategy if he/she chooses her/his first coin with some probability over many runs. We define x , y , and z to be the probabilities of choosing the first coin by Alice, Bob, and Chris, respectively.

Note that the quantities x , y , and z , though being mathematically similar in three- and six-coin setups, are physically different in the following sense. In three-coin setup does not require many runs for the pure-strategy game. Whereas in the six-coin setup many runs are required for both the 'pure strategy' and the 'mixed strategy' games.

In six-coin setup one can define the individual coin probabilities as

$$\begin{aligned} r &= \Pr(+1; S_1), \quad r' = \Pr(+1; S'_1), \quad r'' = \Pr(+1; S''_1), \\ s &= \Pr(+1; S_2), \quad s' = \Pr(+1; S'_2), \quad s'' = \Pr(+1; S''_2), \end{aligned} \quad (11)$$

then factorizability of joint probabilities is expressed, for example, as

$$\Pr(+1, -1, -1; S_2, S'_1, S''_2) = s(1 - r')(1 - s''). \quad (12)$$

We define pure-strategy payoffs by the expressions like

$$\begin{aligned} \Pi_{A,B,C}(S_2, S'_1, S''_1) &= (\alpha, \alpha, \alpha)sr'r'' + (\delta, \beta, \delta)s(1 - r')r'' + (\delta, \delta, \beta)sr'(1 - r'') + \\ &+ (\epsilon, \theta, \theta)s(1 - r')(1 - r'') + (\beta, \delta, \delta)(1 - s)r'r'' + (\theta, \theta, \epsilon)(1 - s)(1 - r')r'' + \\ &+ (\theta, \epsilon, \theta)(1 - s)r'(1 - r'') + (\omega, \omega, \omega)(1 - s)(1 - r')(1 - r''), \end{aligned} \quad (13)$$

while the mixed-strategy payoffs are

$$\begin{aligned} \Pi_{A,B,C}(x, y, z) &= xyz\Pi_{A,B,C}(S_1, S'_1, S''_1) + x(1 - y)z\Pi_{A,B,C}(S_1, S'_2, S''_1) + \\ &+ xy(1 - z)\Pi_{A,B,C}(S_1, S'_1, S''_2) + x(1 - y)(1 - z)\Pi_{A,B,C}(S_1, S'_2, S''_2) + \\ &+ (1 - x)yz\Pi_{A,B,C}(S_2, S'_1, S''_1) + (1 - x)(1 - y)z\Pi_{A,B,C}(S_2, S'_2, S''_1) + \\ &+ (1 - x)y(1 - z)\Pi_{A,B,C}(S_2, S'_1, S''_2) + (1 - x)(1 - y)(1 - z)\Pi_{A,B,C}(S_2, S'_2, S''_2). \end{aligned} \quad (14)$$

A triple (x^*, y^*, z^*) is a NE when

$$\begin{aligned}\Pi_A(x^*, y^*, z^*) - \Pi_A(x, y^*, z^*) &\geq 0, \\ \Pi_B(x^*, y^*, z^*) - \Pi_B(x^*, y, z^*) &\geq 0, \\ \Pi_C(x^*, y^*, z^*) - \Pi_C(x^*, y^*, z) &\geq 0.\end{aligned}\tag{15}$$

3.3 Playing the Prisoner's Dilemma

Consider playing three-player Prisoner's Dilemma (PD) using the three-coin setup. The triple $(x^*, y^*, z^*) = (0, 0, 0) \sim (D, D, D)$ comes out as the unique NE at which the three players are rewarded as $\Pi_A(0, 0, 0) = \Pi_B(0, 0, 0) = \Pi_C(0, 0, 0) = \omega$. In six-coin setup we analyze this game when $(s, s', s'') = (0, 0, 0)$, saying that the probability of getting head from each player's second coin is zero. This reduces the Nash inequalities (15) to

$$\begin{aligned}(x^* - x) \{y^* z^* (rr' r'') \Delta_1 + r(z^* r'' + y^* r') \Delta_2 + r \Delta_3\} &\geq 0, \\ (y^* - y) \{x^* z^* (rr' r'') \Delta_1 + r'(z^* r'' + x^* r) \Delta_2 + r' \Delta_3\} &\geq 0, \\ (z^* - z) \{x^* y^* (rr' r'') \Delta_1 + r''(y^* r' + x^* r) \Delta_2 + r'' \Delta_3\} &\geq 0,\end{aligned}\tag{16}$$

where $\Delta_1 = (\alpha - \beta - 2\delta + 2\theta + \epsilon - \omega)$, $\Delta_2 = (\delta - \epsilon - \theta + \omega)$, and $\Delta_3 = (\epsilon - \omega)$. Now for PD we have $\Delta_3 < 0$ and (D, D, D) comes out the unique NE. This is described by saying that when $(s, s', s'') = (0, 0, 0)$ and the joint probabilities are factorizable the triple (D, D, D) comes out as the unique NE.

We notice that the requirement $(s, s', s'') = (0, 0, 0)$ can also be translated as constraints on the joint probabilities involved in the six-coin setup. For this we first denote these joint probabilities as

$$\begin{aligned}p_1 &= rr' r'', & p_{33} &= rs' s'', & p_{61} &= (1-s)s' s'', \\ p_2 &= r(1-r')r'', & p_{34} &= r(1-s')s'', & p_{62} &= (1-s)(1-s')s'', \\ p_3 &= rr'(1-r''), & \cdots & p_{35} &= rs'(1-s''), & \cdots & p_{63} &= (1-s)s'(1-s''), \\ p_4 &= r(1-r')(1-r''), & p_{36} &= r(1-s')(1-s''), & p_{64} &= (1-s)(1-s')(1-s''),\end{aligned}\tag{17}$$

which allows us to re-express the payoff relations (13) as

$$\begin{aligned}\Pi_{A,B,C}(S_2, S'_1, S''_1) &= (\alpha, \alpha, \alpha)p_9 + (\delta, \beta, \delta)p_{10} + (\delta, \delta, \beta)p_{11} + \\ &(\epsilon, \theta, \theta)p_{12} + (\beta, \delta, \delta)p_{13} + (\theta, \theta, \epsilon)p_{14} + (\theta, \epsilon, \theta)p_{15} + (\omega, \omega, \omega)p_{16} \text{ etc.}\end{aligned}\tag{18}$$

Now, in the six-coin setup, the requirement $(s, s', s'') = (0, 0, 0)$ makes thirty-seven joint probabilities to vanish:

$$p_{(9,10,11,12,17,19,21,23,25,26,29,30,33,34,35,37,38,39,41,42,43,44,45,46,49,50,51,52,53,55,57,58,59,60,61,62,63)} = 0,\tag{19}$$

which simplifies the pure-strategy payoff relations (18) to

$$\begin{aligned}
\Pi_{A,B,C}(S_1, S'_1, S''_1) &= (\alpha, \alpha, \alpha)p_1 + (\delta, \beta, \delta)p_2 + (\delta, \delta, \beta)p_3 + \\
&(\epsilon, \theta, \theta)p_4 + (\beta, \delta, \delta)p_5 + (\theta, \theta, \epsilon)p_6 + (\theta, \epsilon, \theta)p_7 + (\omega, \omega, \omega)p_8; \\
\Pi_{A,B,C}(S_2, S'_1, S''_1) &= (\beta, \delta, \delta)p_{13} + (\theta, \theta, \epsilon)p_{14} + (\theta, \epsilon, \theta)p_{15} + (\omega, \omega, \omega)p_{16}; \\
\Pi_{A,B,C}(S_1, S'_2, S''_1) &= (\delta, \beta, \delta)p_{18} + (\epsilon, \theta, \theta)p_{20} + (\theta, \theta, \epsilon)p_{22} + (\omega, \omega, \omega)p_{24}; \\
\Pi_{A,B,C}(S_1, S'_1, S''_2) &= (\delta, \delta, \beta)p_{27} + (\epsilon, \theta, \theta)p_{28} + (\theta, \epsilon, \theta)p_{31} + (\omega, \omega, \omega)p_{32}; \\
\Pi_{A,B,C}(S_1, S'_2, S''_2) &= (\epsilon, \theta, \theta)p_{36} + (\omega, \omega, \omega)p_{40}; \\
\Pi_{A,B,C}(S_2, S'_1, S''_2) &= (\theta, \epsilon, \theta)p_{47} + (\omega, \omega, \omega)p_{48}; \\
\Pi_{A,B,C}(S_2, S'_2, S''_1) &= (\theta, \theta, \epsilon)p_{54} + (\omega, \omega, \omega)p_{56}; \\
\Pi_{A,B,C}(S_2, S'_2, S''_2) &= (\omega, \omega, \omega)p_{64}.
\end{aligned} \tag{20}$$

These payoff relations ensure that for factorizable joint probabilities the classical outcome of the game results.

4 Three-player quantum games

We consider three-party EPR setting [2] to play a three-player symmetric game such that each player's two directions of measurement are his/her pure strategies. In analogy with the six-coin setup, this is achieved by establishing the association:

$$\begin{aligned}
S_1, S'_1, S''_1 &\sim \text{choose the first direction,} \\
S_2, S'_2, S''_2 &\sim \text{choose the second direction.}
\end{aligned} \tag{21}$$

Now, in a run, each player chooses one direction out of the two and the referee is informed about players' choices. The referee rotates Stern-Gerlach type apparatus along the three chosen directions and performs (quantum) measurement, the outcome of which, along all the three directions, is either +1 or -1.

Comparing the three-party EPR setting to the six-coin setup shows that in a run, choosing between two directions of measurement is similar to choosing between the two coins. The outcome of (quantum) measurement is +1 or -1 as it is the case with the coins.

We now denote the joint probabilities in the three-party EPR setting as

$$\begin{aligned}
p_1 &= \Pr(+1, +1, +1; S_1, S'_1, S''_1), & p_{61} &= \Pr(-1, +1, +1; S_2, S'_2, S''_2), \\
p_2 &= \Pr(+1, -1, +1; S_1, S'_1, S''_1), & p_{62} &= \Pr(-1, -1, +1; S_2, S'_2, S''_2), \\
p_3 &= \Pr(+1, +1, -1; S_1, S'_1, S''_1), & p_{63} &= \Pr(-1, +1, -1; S_2, S'_2, S''_2), \\
p_4 &= \Pr(+1, -1, -1; S_1, S'_1, S''_1), & p_{64} &= \Pr(-1, -1, -1; S_2, S'_2, S''_2),
\end{aligned} \tag{22}$$

which for coins are reduced to the factorizable joint probabilities involved in the six-coin setup.

Quantum mechanics imposes constraints on the joint probabilities involved in the three-party EPR setting. These are usually known as the normalization and the causal communication constraints [5]. Normalization says that

$$\sum_{i=1}^8 p_i = 1, \sum_{i=9}^{16} p_i = 1, \dots \sum_{i=57}^{64} p_i = 1. \quad (23)$$

While, the causal communication constraint is expressed as

$$\begin{aligned} \sum_{i=1}^4 p_i &= \sum_{i=17}^{20} p_i = \sum_{i=25}^{28} p_i = \sum_{i=33}^{36} p_i \\ \sum_{i=5}^8 p_i &= \sum_{i=21}^{24} p_i = \sum_{i=29}^{32} p_i = \sum_{i=37}^{40} p_i \\ \sum_{i=9}^{12} p_i &= \sum_{i=41}^{44} p_i = \sum_{i=49}^{52} p_i = \sum_{i=57}^{60} p_i \\ \sum_{i=13}^{16} p_i &= \sum_{i=45}^{48} p_i = \sum_{i=53}^{56} p_i = \sum_{i=61}^{64} p_i \end{aligned} \quad (24)$$

$$\begin{aligned} \sum_{i=1}^4 p_{2i-1} &= \sum_{i=5}^8 p_{2i-1} = \sum_{i=13}^{16} p_{2i-1} = \sum_{i=21}^{24} p_{2i-1} \\ \sum_{i=1}^4 p_{2i} &= \sum_{i=5}^8 p_{2i} = \sum_{i=13}^{16} p_{2i} = \sum_{i=21}^{24} p_{2i} \\ \sum_{i=9}^{12} p_{2i-1} &= \sum_{i=17}^{20} p_{2i-1} = \sum_{i=25}^{28} p_{2i-1} = \sum_{i=29}^{32} p_{2i-1} \\ \sum_{i=9}^{12} p_{2i} &= \sum_{i=17}^{20} p_{2i} = \sum_{i=25}^{28} p_{2i} = \sum_{i=29}^{32} p_{2i} \end{aligned} \quad (25)$$

$$\begin{aligned} p_1 + p_2 + p_5 + p_6 &= p_{17} + p_{18} + p_{21} + p_{22} = p_9 + p_{10} + p_{13} + p_{14} = p_{49} + p_{50} + p_{53} + p_{54}, \\ p_3 + p_4 + p_7 + p_8 &= p_{19} + p_{20} + p_{23} + p_{24} = p_{11} + p_{12} + p_{15} + p_{16} = p_{51} + p_{52} + p_{55} + p_{56}, \\ p_{25} + p_{26} + p_{29} + p_{30} &= p_{33} + p_{34} + p_{37} + p_{38} = p_{41} + p_{42} + p_{45} + p_{46} = p_{57} + p_{58} + p_{61} + p_{62}, \\ p_{27} + p_{28} + p_{31} + p_{32} &= p_{35} + p_{36} + p_{39} + p_{40} = p_{43} + p_{44} + p_{47} + p_{48} = p_{59} + p_{60} + p_{63} + p_{64}. \end{aligned} \quad (26)$$

Essentially, these constraints state that, in a run, on referee's measurement, the outcome of +1 or -1 along Alice's chosen direction is independent of what choices Bob and Chris make for their directions. The same applies for Bob and Chris.

Notice that the factorizable joint probabilities also satisfy the causal communication constraint as do the three-party EPR joint probabilities. Whereas, unlike the coin probabilities, the three-party EPR joint probabilities can be non-factorizable. In this case if (22) are expressed as (17) one of more of the probabilities r, r', r'', s, s', s'' becomes negative or greater than one.

4.1 Three-player quantum Prisoner's Dilemma

Notice that the constraints (19) ensure that for a factorizable joint probabilities the triple (D, D, D) becomes a NE and that requiring that a set of (quantum mechanical) joint probabilities to satisfy the constraints (19) imbeds the classical game within the corresponding quantum game.

We now consider playing the three-player PD using the three-party EPR setting. We ask whether the triple (C, C, C) can be a NE for non-factorizable joint probabilities while our setup ensures that for factorizable three-party EPR joint probabilities the game can be interpreted classically, with the triple (D, D, D) being its unique NE. To answer this we use (20), (23), and (24-26) to find the NE from (15) and allow the involved joint probabilities to become non-factorizable.

For two-player PD we have reported that (D, D, D) once again emerges as the unique NE with the same definition of players' strategies and under the requirements that embed the classical game within the quantum.

For three-player PD the situation, however, comes out to be different. The Nash inequalities for the triple (C, C, C) then read

$$\{p_5 + (\alpha/\beta)p_1 - p_{13}\} + (\theta/\beta)\{p_6 + p_7 - p_{14} - p_{15} + (\delta/\theta)(p_2 + p_3)\} + (\omega/\beta)\{p_8 - p_{16} + (\epsilon/\omega)p_4\} \geq 0; \quad (27)$$

$$\{p_2 + (\alpha/\beta)p_1 - p_{18}\} + (\theta/\beta)\{p_4 + p_6 - p_{20} - p_{22} + (\delta/\theta)(p_3 + p_5)\} + (\omega/\beta)\{p_8 - p_{24} + (\epsilon/\omega)p_7\} \geq 0; \quad (28)$$

$$\{p_3 + (\alpha/\beta)p_1 - p_{27}\} + (\theta/\beta)\{p_4 + p_7 - p_{28} - p_{31} + (\delta/\theta)(p_2 + p_5)\} + (\omega/\beta)\{p_8 - p_{32} + (\epsilon/\omega)p_6\} \geq 0; \quad (29)$$

where $\alpha/\beta, \theta/\beta, \delta/\theta, \omega/\beta, \epsilon/\omega < 1$. We find that a set of (quantum) joint probabilities that satisfy the normalization and the causal communication constraints can indeed allow the inequalities (27-29) to be true. For example, take $\alpha/\beta = 9/10, \theta/\beta = 1/100, \delta/\theta = 1/5, \omega/\beta = 1/100, \epsilon/\omega = 9/10$ and assign values to these joint probabilities as $p_1 = 1/10, p_3 = 13/100, p_5 = 16/100, p_6 = 1/10, p_{13} = 14/100, p_{15} = 2/5, p_{18} = 13/100, p_{20} = 1/4, p_{22} = 37/100, p_{27} = 1/5$ which we call as the 'independent probabilities'. Notice that constraints (19) assign zero value to thirty seven joint probabilities out of the remaining ones and using the normalization and causal communication constraints the values assigned to the rest of joint probabilities are then found as $p_2 = 7/50, p_4 = 1/100, p_7 = 3/20, p_8 = 21/100, p_{14} = 9/25, p_{16} = 1/10, p_{24} = 1/4, p_{28} = 9/50, p_{31} = 17/50, p_{32} = 7/25, p_{36} = 19/50, p_{40} = 31/50, p_{47} = 27/50, p_{48} = 23/50, p_{54} = 1/2, p_{56} = 1/2$. With this the above NE inequalities for (C, C, C) reduce to $0.106 \geq 0, 0.096 \geq 0, 0.017 \geq 0$ which are trivially true.

Note that for PD we have $\alpha/\beta, \theta/\beta, \delta/\theta, \omega/\beta$, and ϵ/ω all less than zero and not every non-factorizable set of joint probabilities can result in (C, C, C) being a NE. In this paper we do not explore which other NE may emerge for a given non-factorizable set of probabilities. However, we notice that the classical outcome of (D, D, D) being a NE remains intact even when the joint probabilities may become non-factorizable. This means that a set of non-factorizable joint probabilities can only add to the unique classical NE in the three-player PD.

5 Concluding remarks

We use three-party EPR setting to play a three-player symmetric noncooperative game. Players' payoffs are re-expressed in terms of players' choices in the EPR setting and in terms of the joint probabilities. We use Nash inequalities

in the six-coin setup to impose constraints on joint probabilities which ensure that with factorizable joint probabilities the game has a classical interpretation. We then retain these constraints while allowing the joint probabilities to become non-factorizable and find how non-factorizable probabilities may lead to the emergence of new solutions of the game. Multi-player quantum games are, therefore, constructed in terms of probabilities only and it is shown that non-factorizable joint probabilities may lead to different game-theoretic outcome(s). We find that with this framework it is hard to construct Enk & Pike type argument [6] for a quantum game.

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